ISRAEL JOURNAL OF MATHEMATICS, Vol. 34, No. 3, 1979

ALMOST AUTOMORPHIC SYMBOLIC MINIMAL SETS WITHOUT UNIQUE ERGODICITY

BY

NELSON G. MARKLEY AND MICHAEL E. PAUL

ABSTRACT

Given a metrizable monothetic group G with generator g and a suitable closed nowhere dense subset C of positive Haar measure, we associate a natural compact metric space whose points are almost automorphic symbolic minimal sets. It is then shown that those minimal sets which have positive topological entropy and fail to be uniquely ergodic form a residual set. The example due to P. Julius [2] of a Toeplitz sequence of positive entropy which is uniquely ergodic shows that the "residual" conclusion is sharp.

Around 1972 a number of people conjectured that a compact metric almost automorphic minimal set was uniquely ergodic only if the image of the almost automorphic points in its maximal equicontinuous factor had full measure with respect to the unique invariant probability measure on the latter. Since the converse is trivial, a proof of this conjecture would have provided a simple characterization of unique ergodicity in this setting. However, recently P. Julius [2] has given an example of a Toeplitz sequence whose orbit closure is uniquely ergodic and has positive entropy. Consequently this conjecture is false. The purpose of this paper is to show that in some settings including Toeplitz sequences it is usually true. Specifically there are natural compact metric spaces of almost automorphic symbolic minimal sets whose almost automorphic points do not have full measure in the above sense, and our results give conditions for the existence of a residual collection of these minimal sets which are not uniquely ergodic.

In his seminal paper on disjointness Furstenberg [1] gave an example of a minimal symbolic flow with positive entropy which was not uniquely ergodic. Various people recognized that this example was almost automorphic, in fact Toeplitz. The main idea in the construction of this example is to get an arbitrary sequence of symbols to appear in order but not consecutively in a block whose

Received September 28, 1978

length can be controlled. We make use of this idea from the viewpoint of separating covers or characteristic sequences developed by the authors [4 and 5].

1. Introduction

Let (X, φ) be a minimal (discrete) flow on a compact metric space. There exists a maximal equicontinuous factor of (X, φ). Since the group acting on X is the integers, this maximal equicontinuous factor can be represented as a monothetic group G with translation by a generator g and will be denoted by (G, g). Let p be a homomorphism of (X, φ) onto (G, g). A point $x \in X$ is an almost automorphic point if $p^{-1}(p(x)) = \{x\}$. If (X, φ) is almost automorphic, that is, X contains an almost automorphic point, then $p(x) = p(y)$ if and only if x and y are proximal.

A Borel probability measure μ on X is invariant if $\mu(\varphi^{-1}(A)) = \mu(A)$ for all Borel sets A. At least one such measure always exist; when only one exists (X, φ) is said to be uniquely ergodic. In general the ergodic measures are the extreme points of the convex compact set of invariant measures. For every ergodic measure μ on X there exists an $x \in X$ such that

$$\int_X f du = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(\varphi^k(x))$$

for all continuous real valued functions. If this limit exists for all x in X and all continuous f, then (X, φ) is uniquely ergodic and for each continuous f the convergence is uniform in x.

The full shift on m-symbols will be denoted by (Ω_m, σ). If $x \in \Omega_m$, then $x(j)$ will denote the j-th coordinate of x. We will also need the space Ω_m^+ consisting of all sequences on m-symbols indexed by the positive integers. As usual the symbols will be $\{0, \cdots, m-1\}$.

Let (M, σ) be a minimal subset of (Ω_m, σ). An l-block $B = b_1 \cdots b_l$ appears in M if there exists $x \in M$ and r such that $x(r+1) \cdots x(r+l) = b_1 \cdots b_l$. The (topological) entropy of (M, σ) is given by

$$h(M) = \lim_{N \to \infty} \frac{1}{N} \log |\{B : B \text{ is an } N\text{-block appearing in } M\}|.$$

(Here $|\quad|$ denotes cardinality.)

2. Separating covers

Let G be a compact metrizable monothetic group with generator g. Following

[5] we define a separating cover of (G, g) to be an ordered finite cover $\alpha = \{D_0, \cdots, D_{m-1}\}$ of G satisfying:

(a) $\mathrm{cl}(\mathrm{int}(D_i)) = D_i$ for all i,

(b) $\mathrm{int}(D_i \cap D_j) = \varnothing$ for all $i \neq j$,

(c) if z and w are distinct elements, then there exist integers n, i, j with $i \neq j$ such that $z + ng \in \mathrm{int}\, D_i$ and $w + ng \in \mathrm{int}\, D_j$.

It is easy to check that condition (c) is equivalent to:

(c′) $D_i + z = D_i$ for $i = 0, \cdots, m-1$ implies $z = 0$.

Let (M, σ) be an almost automorphic minimal subset of (Ω_m, σ) and let $p\colon (M, \sigma) \to (G, g)$ be a homomorphism onto its maximal equicontinuous factor. Note G is metric in this case. Set $D_i = p(\{x \in M\colon x(0) = i\})$ for $i = 0, \cdots, m-1$. Then $\alpha = \{D_0, \cdots, D_{m-1}\}$ is a separating cover of (G, g). Letting $\partial\alpha = \bigcup_i \partial D_i = \bigcup_{i \neq j} D_i \cap D_j$ and $U = G \backslash \partial\alpha$ we have $A = \bigcap_{n=-\infty}^{\infty} U + ng$ is residual and $x \in M$ is almost automorphic if and only if $p(x) \in A$. Moreover, when $p(x) = z \in A$,

$$x(n) = \sum_{i=0}^{m-1} i\chi_{D_i}(z + ng). \tag{1}$$

(Throughout χ_W will denote the characteristic function of the set W.) Conversely, if $\alpha = \{D_0, \cdots, D_{m-1}\}$ is a separating cover of (G, g), then (1) can be used to construct an almost automorphic minimal set $(M(\alpha), \sigma)$ in (Ω_m, σ) which is independent of z and a homomorphism $p\colon (M(\alpha), \sigma) \to (G, g)$ such that $D_i = p(\{x \in M(\alpha)\colon x(0) = i\})$ and $p(x) = p(y)$ if and only if x and y are proximal. In particular, (G, g) is the maximal equicontinuous factor of $(M(\alpha), \sigma)$. If (c) does not hold, then (1) still defines an almost automorphic minimal set in (Ω_m, σ) but the maximal equicontinuous factor need not be (G, g). (For additional information about this point of view the reader is referred to [4] and [5].)

Remark 1. Let $\alpha = \{D_0, \cdots, D_{m-1}\}$ be an ordered finite cover of G satisfying conditions (a) and (b). If $\partial\alpha + z = \partial\alpha$ implies $z = 0$, then α is a separating cover.

Proof. If $D_i + z = D_i$ for $i = 0, \cdots, m-1$, then $\partial\alpha + z = \partial\alpha$.

Let λ denote Haar measure on G and note that $\lambda(A)$ is either 1 or 0. When $\lambda(A) = 1$, $(M(\alpha), \sigma)$ is uniquely ergodic. If $\lambda(A) = 0$, then $\lambda(\partial\alpha) > 0$ and U is endowed with some structural complexity. In particular, there may be many separating covers with the same boundary. We will show that in many cases there is a residual subset of such covers for which $(M(\alpha), \sigma)$ is not uniquely ergodic.

The general setting for the remainder of the paper is specified by the following:

Standing Hypothesis

(1) C is a closed nowhere dense subset of a compact metric monothetic group G, and g is a generator of G.

(2) There is a fixed sequence $\{U_i\}_{i=1}^{\infty}$ of disjoint open sets of G satisfying:

(i) $G \backslash C = \bigcup_{i=1}^{\infty} U_i$,

(ii) given a neighborhood V of z an element of C, $V \cap U_i \neq \emptyset$ for infinitely many i's.

Given $\omega \in \Omega_m^+$ set

$$D_i = \mathrm{cl}[\bigcup \{U_j : \omega(j) = i\}]$$

and observe that (a) and (b) hold. Thus each ω gives rise to an almost automorphic symbolic minimal set which will be denoted by $M(\omega)$. By assuming $C + z = C$ implies $z = 0$ we can conclude that $\alpha = \{D_0, \cdots, D_i\}$ is a separating cover of (G, g) provided $\partial\alpha = C$. In general $\partial\alpha$ need not coincide with C, but we do have the following:

LEMMA 2. *If $C + z = C$ implies $z = 0$, then there exists a residual set R_0 in Ω_m^+ such that*

(i) $C = \partial\alpha$,

(ii) *α is a separating cover,*

(iii) *(G, g) is the maximal equicontinuous factor of $(M(\omega), \sigma)$,*

(iv) *if $\omega \in R_0, \omega' \in \Omega_m^+$, and $\omega(n) = \omega'(n)$ for all $n \geqq N$, then $\omega' \in R_0$.*

PROOF. By the preceding remarks it suffices to prove (i) and (iv). Let $z \in C$ and let $d(\ ,\ )$ denote a metric for G. Then $z \notin \partial\alpha$ if and only if there exist $k > 0$ and j, $0 \leqq j \leqq m - 1$ such that $\omega(i) = j$ for all i with $d(U_i, z) < 1/k$. Let

$$P(z, j, k) = \left\{\omega \in \Omega_m^+ : \omega(i) \neq j \ \forall i \geqq k \ni d(U_i, z) < \frac{1}{k}\right\}$$

and let

$$P = \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{m-1} \bigcup_{k=1}^{\infty} P(z_m, j, k),$$

where $\{z_n\}_{n=1}^{\infty}$ is a countable dense subset of C. Since there are infinitely many i's such that $d(U_i, z) < 1/k$ for $z \in C$, $P(z, j, k)$ is closed and nowhere dense. Finally $R_0 = \Omega_m^+ \backslash P$ is the desired set.

3. Basic principles

Continuing with the same notation and assumptions select and fix $z_0 \in G$. Define a sequence $\{c_n\}_{n=1}^{\infty}$ inductively by first letting $c_1 = 1$ and n_1 satisfy $z_0 + g \in U_{n_1}$, and then — assuming $c_1, \cdots, c_k$ and $n_1, \cdots, n_k$ are defined — letting

$$c_{k+1} = \min\left\{j: z_0 + jg \notin \bigcup_{i=1}^{k} U_{n_i}\right\}$$

and n_{k+1} satisfy $z_0 + c_{k+1}g \in U_{n_{k+1}}$. Thus c_n is the minimal number of steps required for $z_0 + kg$, $k \geqq 1$ to hit precisely n distinct U_i's. Note $\{c_n\}_{n=1}^{\infty}$ is an increasing sequence and $i \to n_i$ is a permutation of the positive integers. By reindexing we can assume that $z_0 + c_n g \in U_n$ and for $1 \leqq k < c_n$, $z_0 + kg \in \bigcup_{i=1}^{n-1} U_i$.

THEOREM 3. *If* $\underline{\lim}_{N\to\infty} c_n/n = \delta < \infty$ *and* $m > 2^\rho$ *where* $\rho = [\delta] + 1$, *then there exists a residual set* R *in* Ω_m^+ *such that for all* $\omega \in R$, $M(\omega)$ *is not uniquely ergodic and* $h(M(\omega)) > (1/\rho)\ln(m/2^\rho) > 0$.

PROOF. Let $l \geqq 2$ and define

$$Q_l = \{r: c_{r+l} - c_{r+1} < l\rho\}.$$

The first step in the proof is to show that Q_l is infinite. Assume $|Q_l| = K < \infty$ and consider N such that $c_N \leqq \delta N + 1$. We can write $N = sl + t$ with $0 \leqq t < l$. Then

$$\begin{aligned}(s-K)l\rho &\leqq (c_l - c_1) + (c_{2l} - c_{l+1}) + \cdots + (c_{sl} - c_{(s-1)l+1}) \\ &\leqq \sum_{j=1}^{s-1} (c_{jl} - c_{jl+1}) - c_1 + c_{sl} \leqq c_N \\ &\leqq \delta(sl + t) + 1\end{aligned}$$

and dividing by $(s-K)l$ gives

$$\rho \leqq \frac{\delta s}{s-K} + \frac{\delta t}{(s-K)l} + \frac{1}{(s-K)l}.$$

Since we can let N go to infinity with $c_N \leqq \delta N + 1$, we get the contradiction $\rho \leqq \delta$ because s would also go to infinity.

Let B be any l block on m-symbols and set

$$Q(B) = \{\omega \in \Omega_m^+: \omega(r+1)\cdots\omega(r+l) \neq B \,\forall r \in Q_l\}.$$

It follows that each $Q(B)$ is closed and nowwhere dense in Ω_m^+. Now let

$$R = \Omega_m \setminus \left[\bigcup_B Q(B) \right].$$

It remains to show that for any $\omega \in R$, $M(\omega)$ has positive entropy and is not uniquely ergodic. To do this we will make use of the point $x_0(n) = \sum_{i=0}^{m-1} i\chi_{D_i}(z_0 + ng)$ in $M(\omega)$.

Let $B = b_1 \cdots b_l$ be any l-block in m symbols. There exists $r \in Q_l$ such that $\omega(r+1) \cdots \omega(r+l) = B$ because $\omega \in R$. Since $c_{r+l} < c_{r+1} + l\rho$ and $x(c_n) = \omega(n)$ for all n, the symbols $b_1, \cdots, b_l$ appear in order but not necessarily consecutively in $x(c_{r+1}) \cdots x(c_{r+1} + l\rho - 1)$. But a ρl-block can produce at most $\binom{\rho l}{l}$ l-blocks in this way. Therefore

$$m^l \leqq \binom{\rho l}{l} \left| \{A : A \quad \text{is a} \quad \rho l\text{-block in} \quad M(\omega)\} \right|.$$

Because $\binom{\rho l}{l} \leqq 2^{\rho l}$ the above implies

$$(m/2^\rho)^l \leqq \left| \{A : A \quad \text{is a} \quad \rho l\text{-block in} \quad M(\omega)\} \right|$$

and consequently

$$0 < \frac{\log (m/2^\rho)}{\rho} \leqq h(M(\omega)).$$

If $M(\omega)$ were uniquely ergodic, then each symbol j would have a frequency γ_j and $\sum_{j=0}^{m-1} \gamma_\gamma = 1$. Using the previous argument we can find a ρl-block A in x containing at least l j's. Thus $\gamma_j \geqq 1/\rho$ and the contradiction $1 \geqq m/\rho > 1$ completes the proof.

Since $M(\omega)$ is uniquely ergodic if $\omega \in R_0$ and $m(C) = 0$ and since $R \cap R_0 \neq \emptyset$, $\underline{\lim}\, c_n/n < \delta$ implies $m(C) > 0$.

Furstenberg's example [1] of a minimal set with positive entropy which is not uniquely ergodic is the prototype of the previous theorem. The next two theorems are variations on the same theme which produce the same results without the restriction on the number of symbols.

THEOREM 4. *If $\underline{\lim}\, c_n/n = \delta < \infty$, $\lambda(\partial U_i) = 0$ for all i, and $m \geqq 2$, then there exists a residual set R in Ω_m^+ such that $M(\omega)$ is not uniquely ergodic for all $\omega \in R$.*

PROOF. First take $m = 2^\rho + 1$ and apply the previous theorem to obtain an $\bar{\omega}$ such that $M(\bar{\omega})$ is not uniquely ergodic. Let $V_i = \{\bar{x} \in M(\bar{\omega}) : \bar{x}(0) = i\}$ and suppose that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_{V_i}(\sigma^k(\bar{x})) = \gamma_i(\bar{x})$$

exists for all $\tilde{x} \in M(\tilde{\omega})$ and $0 \leqq i \leqq m-1$. Then each γ_i is invariant and has a residual set of points of continuity because it is a pointwise limit of continuous real valued functions on a Baire space. By minimality each $\gamma_i(\tilde{x})$ is a constant γ_i. Thus for every ergodic measure μ we have $\mu(V_i) = \gamma_i$ which in turn implies the same thing for all invariant measures. From the choice of $\tilde{\omega}$ we know that for each i there exists an invariant measure such that $\mu(V_i) \geqq 1/\rho$ where $\rho = [\delta] + 1$. Now we again have the contradiction $1 = \sum_{i=0}^{m-1} \gamma_i \geqq m/\rho > 1$. Therefore, there exists $\tilde{x}_1 \in M(\tilde{\omega})$ and i such that

$$\lim_{N \to 0} \frac{1}{N} \sum_{k=0}^{N-1} \chi_{V_i}(\sigma^k(\tilde{x}_i))$$

does not exist. Without loss of generality we can assume $i = 1$.

The next step is to show that we can assume $\tilde{x}_1$ is almost automorphic. Define f_N and g_K on $M(\tilde{\omega})$ by

$$f_N(\tilde{x}) = \frac{1}{N} \sum_{k=0}^{N-1} \chi_{V_1}(\sigma^k(\tilde{x}))$$

and

$$g_K(\tilde{x}) = \sup_{N, N' \geqq K} |f_N(\tilde{x}) - f_{N'}(\tilde{x})|.$$

Set $V(K, n) = \{\tilde{x} \colon g_K(\tilde{x}) < 1/n\}$ and choose n' so that

$$\overline{\lim} f_N(\tilde{x}_1) - \underline{\lim} f_N(\tilde{x}_1) > 3/n'.$$

Suppose $\operatorname{int}(\operatorname{cl} V(K, n')) = W \neq \emptyset$. Then there exists η such that $\tilde{x}_2 = \sigma^\eta(\tilde{x}_1) \in W$. Choose $N > N' > K$ so that $f_N(\tilde{x}_2) - f_{N'}(\tilde{x}_2) > 1/n'$, and choose $\tilde{x} \in V(K, n')$ so that $\tilde{x}(j) = \tilde{x}_2(j)$ for $0 \leqq j \leqq N$. Then we have the contradiction

$$1/n' > g_K(\tilde{x}) \geqq f_N(\tilde{x}) - f_{N'}(\tilde{x}) = f_N(\tilde{x}_2) - f_{N'}(\tilde{x}_2) > 1/n'.$$

It follows that $M(\tilde{\omega}) \setminus \bigcup_{K=1}^{\infty} \operatorname{cl} V(K, n')$ is a residual subset of $M(\tilde{\omega})$ and hence contains an almost automorphic point. Since $\lim_{N \to \infty} f_N(\tilde{x})$ exists if and only if $\tilde{x} \in \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} V(K, n)$, we can assume that $\tilde{x}_1$ is almost automorphic.

Let $z_1 = p(\tilde{x}_1)$ and reindex $\{U_n\}_{n=1}^{\infty}$ using z_1 instead of z_0. (We will not be using the hypothesis that $\underline{\lim}\, c_n/n = \delta < \infty$ again.) This yields a new $\tilde{\omega}_1$ such that $M(\tilde{\omega}_1) = M(\tilde{\omega})$. Consider the block map $0, 2, \cdots, m-1 \to 0$ and $1 \to 1$. It defines a map of Ω_m^+ onto Ω_2^+ and a homomorphism $\varphi \colon M(\tilde{\omega}_1) \to \Omega_2$. Let ω_1 be the image of $\tilde{\omega}_1$ and x_1 the image of $\tilde{x}_1$. It is obvious that the image of $M(\tilde{\omega}_1)$ is $M(\omega_1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} x_1(k)$$

does not exist, and

$$x_1(n) = \chi_{D_1}(z_1 + ng)$$

where $D_1 = \mathrm{cl}[\bigcup \{U_j : \omega_1(j) = 1\}]$. Let $f: \Omega_2^+ \to \Omega_2$ by

$$f(\omega)(n) = \chi_{D_1}(z_1 + ng)$$

where $D_1 = \mathrm{cl}[\bigcup \{U_j : \omega(j) = 1\}]$. So $f(\omega_1) = x_1$. Set

$$f_N(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} f(\omega)(k) \quad \text{and} \quad g_K(\omega) = \sup_{N,N' \geqq K} |f_N(\omega) - f_{N'}(\omega)|.$$

Suppose $\omega_2(i) = \omega_1(i)$ for all i except i' and $\omega_2(i') \neq \omega_1(i')$. Let $y(k) = \chi_{U_{i'}}(z_1 + kg)$. Because $\lambda(\partial U_i) = 0$, $\lim_{N\to\infty}(1/N)\Sigma_{k=0}^{N-1} y(k)$ exists. Since $f(\omega_2) = f(\omega_1) + y$ or $f(\omega_2) + y = f(\omega_1)$,

$$\overline{\lim} f_N(\omega_2) - \underline{\lim} f_N(\omega_2) = \overline{\lim} f_N(\omega_1) - \underline{\lim} f_N(\omega_1) > 0.$$

Using this ability to change a finite number of coordinates of ω_1 instead of minimality, we can repeat the earlier argument to show that $V(K, n') = \{\omega : g_K(\omega) < 1/n'\}$ is nowhere dense for a suitable n'. Therefore, $R = \Omega_2^+ \setminus \bigcup_{K=1}^{\infty} \mathrm{cl}\, V(K, n')$ is a residual set such that $\lim_{N\to\infty} f_N(\omega)$ does not exist for all $\omega \in R$. In particular, $M(\omega)$ is not uniquely ergodic when $\omega \in R$. This proves the theorem for $m = 2$, which implies the result for $2 < m < 2^p + 1$.

THEOREM 5. *If* $\underline{\lim}\, c_n/n = \delta < \infty$ *and* $m \geqq 2$, *then there exists a residual set* R *in* Ω_m^+ *such that for all* $\omega \in R$, $h(M(\omega)) \geqq (1/\rho)\log m$ *where* $\rho = [\delta] + 1$.

PROOF. With Q_l as in the proof of Theorem 3 define $F_l : Q_l \to \{1, \cdots, \rho l\}^{l-1}$ by

$$F_l(r) = (c_{r+2} - c_{r+1}, \cdots, c_{r+l} - c_{r+l-1}).$$

There exists η such that $Q_l' = F_l^{-1}(\eta)$ is infinite. For an l-block B let

$$Q'(B) = \{\omega : \omega(r+1) \cdots \omega(r+l) \neq B \,\forall r \in Q_l'\}$$

and let

$$R = \Omega_m^+ \setminus \bigcup_B Q'(B).$$

Consider l and let $\xi_j = \Sigma_{i=1}^{j-1} \eta_i$ where $\eta = (\eta_1, \cdots, \eta_{l-1})$. For each l-block $B = b_1 \cdots b_l$ there exists a ρl-block A in x containing the symbols of B in order but not consecutively as before, but, in addition, $a_1 = b_1$, $a_{1+\xi_1} = b_2, \cdots, a_{1+\xi_l} = b_l$. Therefore,

$$m^l \leqq |\{A : A \quad \text{is a} \quad \rho l\text{-block in} \quad M(\omega)\}|$$

and

$$\frac{\log m}{\rho} \leqq h(M(\omega)).$$

A variety of corollaries can be obtained by intersecting the residual sets in Lemma 2 and Theorems 3, 4, and 5. Implicit in these results is the topologizing of a certain class of minimal sets. *A priori* this topology depends upon z_0, but it can be easily shown that this does not happen. Hence the residual sets in these results are of some intrinsic interest. It is also true that these residual sets have full measure with respect to the fair Bernoulli measure on Ω_m^+. The remainder of the paper is devoted to showing that these theorems are broadly applicable. In particular, they apply to Cantor sets of positive Lebesgue measure on the circle.

4. Applications

4.1. *p-adic Integers*

A necessary and sufficient condition for a symbolic minimal flow to be the orbit-closure of a Toeplitz bisequence is that the flow be almost automorphic with a totally disconnected compact metric monothetic group as maximal equicontinuous factor. To see how the preceding results apply in this setting, let G_p denote the p-adic integers, with generator 1. Now let $\{n_i\}$ be any increasing sequence of positive integers (if $p = 2$, we require that $n_1 \geqq 2$) and consider the closed and open subgroups $\{H_{n_i}\}$, where

$$H_{n_i} = \text{cl}\{lp^{n_i} : l = 0, \pm 1, \pm 2, \cdots\}.$$

Define $\{U_n\}$ by $U_1 = 1 + H_{n_1}$ and in general $U_n = m + H_{n_k}$, where $m = \inf\{l > 0 : l \notin \bigcup_{j=1}^{n-1} U_j\}$ and $n_k = \inf\{n_i : [m + H_{n_i}] \cap [\bigcup_{j=1}^{n-1} U_j] = \varnothing$ and $n_i > n_s$, where $U_{n-1} = t + H_{n_s}\}$. Let $C = G_p \setminus \bigcup_{n=1}^{\infty} U_n$. Then C is a closed set with empty interior, each U_n has empty boundary, and the Haar measure of C is greater than or equal to

$$1 - \sum_{n=1}^{\infty} p^{-n} = \frac{p-2}{p-1} > 0$$

(if $p=2$, we have required C to have measure greater than or equal to $1-\Sigma_{n=2}^{\infty}2^{-n}=\frac{1}{2}$). Suppose now that there exists a $z\neq 0$ such that $C+z=C$. Then $C+H=C$, where H is the subgroup defined by $H=\mathrm{cl}\{lz: l=0,\pm 1,\pm 2,\cdots\}$. Each nontrivial closed subgroup of G_p is also open, and thus $C+H=C$ implies that C is open, a contradiction. Finally, note that if $x\in C$ and V is a neighborhood of x, then $V\cap U_i\neq\emptyset$ for infinitely many i's, as the boundary of each U_n is empty. Letting $z_0=0$, the only condition which needs to be checked in order to apply the general theorems from Section 3 is the condition that $\underline{\lim}\, c_n/n<\infty$. To see this, let a positive integer n be fixed. Then since U_j is a coset of a subgroup with index $\lambda(U_j)^{-1}$, λ = Haar measure,

$$c_n \leqq c_n\left(\sum_{j=1}^{n-1}\lambda(U_j)\right)+n.$$

The n on the right hand side of the inequality accounts for the starting points $\{c_1,c_2,\cdots,c_n\}$. Thus

$$c_n\lambda(C)\leqq n \quad\text{and}\quad \overline{\lim}\, c_n/n\leqq\lambda(C)^{-1}<\infty.$$

The preceding construction started with G_p and produced symbolic minimal flows which are orbit-closures of Toeplitz bisequences. The connection between this viewpoint, Furstenberg's example [1], and the usual arithmetic progression definition of Toeplitz sequences is given by the following theorem. This theorem asserts that "non-regular Toeplitz sequences usually have positive entropy and are usually not uniquely ergodic." We suppose that $\mathbf{Z}^+=\bigcup_{n=1}^{\infty}P_n$, where $P_n=\{c_n+ld_n: l=0,1,2,\cdots\}$ and $c_1<c_2<\cdots$. For $\omega\in\Omega_m^+$, let $T(\omega)\in\Omega_m^+$ be defined by $T(\omega)_i=\omega_n$, where $i\in P_n$.

Theorem 6. *Suppose that the arithmetic progressions P_n have lengths d_n satisfying $\Sigma\, 1/d_n=\varepsilon<1$. Then for all $m\geqq 2$, there exists a residual set $R\subseteq\Omega_m^+$ such that for each $\omega\in R$, the orbit-closure of $T(\omega)$ has positive topological entropy and is not uniquely ergodic.*

Proof. The theorems of Section 3 apply here if one can show that $\underline{\lim}\, c_n/n<\infty$. To see this, argue exactly as in the preceding construction. Let n be fixed. Then

$$c_n\leqq c_n\left(\sum_{j=1}^{n-1}1/d_j\right)+n,$$

and thus $\overline{\lim}\, c_n/n\leqq(1-\varepsilon)^{-1}$.

4.2. *Tori*

Let G be an l-dimensional torus ($l \leqq \aleph_0$) with generator g. Define a collection of open disks $\{U_n\}$ by first letting U_1 be an open l-disk containing g, $\{ng\}_{n=-\infty}^{\infty} \cap \partial(U_1) = \emptyset$, and such that $\lambda(U_1) < \frac{1}{3}$. Since $\lambda(\partial U_1) = 0$, it follows that for all sufficiently large N,

$$\operatorname{card}([1, N] \cap \{j : jg \in U_1\}) < N/2.$$

Let N_1 be such an N, and then define pairwise disjoint open disks U_i, $2 \leqq i \leqq 1 + (N_1 - \operatorname{card}([1, N] \cap \{j : jg \in U_1\})$ so that:

(1) $U_i \cap U_1 = \emptyset$,
(2) $\lambda(U_i) < 1/3^i$,
(3) $ng \notin \partial(U_i)$ for all n,
(4) $\operatorname{card}([1, N_1] \cap \{j : jg \in U_i\}) = 1$.

Thus if $n_1 = 1 + (N_1 - \operatorname{card}([1, N_1] \cap \{j : jg \in U_1\}))$ and $c_{n_1} = \min\{j : jg \in U_{n_1}\}$, then

$$N_1/2 \leqq n_1, \quad c_{n_1} \leqq N_1, \qquad \text{and so} \quad c_{n_1}/n_1 \leqq 2.$$

Now find $N_2 > N_1$ such that

$$\operatorname{card}\left([1, N_2] \cap \left\{j : jg \in \bigcup_{s=1}^{n_1} U_i\right\}\right) < N_2/2.$$

Define pairwise disjoint open disks U_i,

$$n_1 < i \leqq n_1 + \left(N_2 - \operatorname{card}\left([1, N_2] \cap \left\{j : jg \in \bigcup_{s=1}^{n_1} U_s\right\}\right)\right)$$

so that

(1) $U_i \cap U_s = \emptyset$, $s \leqq n_1$,
(2) $\lambda(U_i) < 1/3^i$,
(3) $ng \notin \partial(U_i)$ for all n,
(4) $\operatorname{card}([1, N_2] \cap \{j : jg \in U_i\}) = 1$.

Then if $n_2 = n_1 + N_2 - \operatorname{card}([1, N_2] \cap \{j : jg \in \bigcup_{s=1}^{n_1} U_s)$,

$$N_2/2 \leqq n_2, \quad c_{n_2} \leqq N_2, \qquad \text{and so} \quad c_{n_2}/n_2 \leqq 2.$$

Continue this process, defining $\{U_n\}_{n=1}^{\infty}$. Without loss of generality we may assume that no two of the disks have the same radius, and that the disks have pairwise disjoint closures. Let $z_0 = 0$ and $C = G \setminus \bigcup U_n$. Then C is a closed set with empty interior, $\lambda(\partial U_n) = 0$, and $\underline{\lim}\, c_n/n \leqq 2$. We need to check two conditions in order to apply the general theorems from Section 3:

(α) $C + z = C$ implies $z = 0$,

(β) if $x \in C$ and V is a neighborhood of x, then $V \cap U_i \neq \emptyset$ for infinitely many i's.

The geometry of the torus implies both conditions. If $z \neq 0$ and $C + z = C$, then there is an index set I with cardinality at least two such that

$$U_1 = \bigcup_{i \in I} (U_i + z).$$

Since $U_i \cap U_j = \emptyset$, it follows that $(U_i + z) \cap (U_j + z) = \emptyset$, and thus $U_1 = \bigcup_{i \in I} (U_i + z)$ violates the connectedness of U_1. To check condition (β) note that the only possibly troublesome points $x \in C$ are those which occur in the boundary of some U_i. Since the U_n were chosen to have pairwise disjoint closures, we assume that $x \in \partial U_i$ and $x \notin \mathrm{cl}(\bigcup_{j \neq i} U_j)$. This implies that $\mathrm{int}(\mathrm{cl}\, U_i) \setminus U_i \neq \emptyset$. To avoid this, we now impose another condition on the construction of the U_n; namely that each U_n satisfies $\mathrm{int}(\mathrm{cl}\, U_n) \setminus U_n = \emptyset$. One can always find such U_n on a torus, and therefore condition (β) must hold for such U_n.

4.3. *The Circle*

The examples constructed on p-adics and tori were fairly specific — one had to be careful in particular to guarantee that $\underline{\lim}\, c_n/n < \infty$. The situation is much more general on the circle $G = \mathbf{R}/\mathbf{Z}$ with irrational rotation γ. Let C be any Cantor set in G, and write $C = G \setminus \bigcup_{n=1}^{\infty} U_n$, where the U_n are the complementary intervals of C. Even though there may exist a finite set of $z \neq 0$ such that $C + z = C$, it can be shown that there is a residual set of 2-symbol ω such that $(M(\omega), \sigma)$ has (G, γ) as maximal equicontinuous factor and $C = \partial\gamma$. The only condition to verify in order to apply the theorems of Section 3 is covered by the following lemma.

Lemma 7. *Let* $\lambda(C) > 0$ *and* $z_0 \in \bigcap_{-\infty}^{\infty} [(G \setminus C) + n\gamma]$. *Then* $\underline{\lim}\, c_n/n \leqq 2\lambda(C)^{-1}$.

Proof. The proof rests on the continued fraction expansion of the irrational number γ. First define the numbers D_n for $n \geqq 1$ by

$$D_n = \sup_{0 \leqq \alpha < \beta \leqq 1} \left| \mathrm{card}\{i : 1 \leqq i \leqq n, \alpha \leqq z_0 + i\gamma < \beta\}/n - (\beta - \alpha) \right|.$$

It is shown in Kuipers and Niederreiter [3, proof of theorem 3.4, ch. 2] that for the sequence $q_1 < q_2 < \cdots$ of denominators of the convergents to γ,

$$D_{q_n} \leqq 1/q_n + 1/q_{n+1},$$

and thus $q_n D_{q_n} < 2$.

Now fix q_n and use the preceding inequality to deduce that

$$q_n \leqq q_n[1-\lambda(C)] + lq_nD_{q_n} \leqq q_n[1-\lambda(C)] + 2l,$$

where $l = \max\{i\colon c_i \leqq q_n\}$. Thus

$$c_l/l \leqq q_n/l \leqq 2\lambda(C)^{-1}.$$

As $q_n \to \infty$, then $l \to \infty$ and so $\underline{\lim}\, c_n/n \leqq 2\lambda(C)^{-1}$. The following theorem is thus immediate.

THEOREM 8. *Let C be a Cantor subset of $\mathbf{R}/\mathbf{Z}$ such that $\lambda(C) > 0$. Then for any irrational number γ there is a residual set $R \subseteq \Omega_2^+$ such that for any $\omega \in R$:*

(1) *$(M(\omega), \sigma)$ is almost automorphic,*

(2) *$(M(\omega), \sigma)$ is not uniquely ergodic,*

(3) *$(M(\omega), \sigma)$ has positive topological entropy,*

(4) *$\partial\alpha = C$,*

(5) *$(\mathbf{R}/\mathbf{Z}, \gamma)$ is the maximal equicontinuous factor of $(M(\omega), \sigma)$.*

EXAMPLES. Suppose now that $(M(\omega), \sigma)$ comes from the preceding theorem. It is then possible to take an almost automorphic point $x \in M(\omega)$ and "split" it into a doubly asymptotic pair y, z such that $x(i) = y(i) = z(i)$ for all $i \neq 0$. Denote the new flow by (M', σ). It follows that (M', σ) is not uniquely ergodic and has the same entropy as $(M(\omega), \sigma)$. Now let (M'', σ) be the result of doubling the ones in (M', σ). Then (M'', σ) has the following properties:

(1) (M'', σ) is weakly mixing,

(2) (M'', σ) is not uniquely ergodic,

(3) (M'', σ) has positive topological entropy,

(4) (M'', σ) is not a prime flow.

Properties (2) and (3) are straightforward; (1) follows from Petersen and Shapiro [6, theorem 2.2]; and (4) follows from the fact that any symbolic flow has nontrivial (i.e., not equal to a fixed point) factors of arbitrarily small topological entropy. Thus any prime symbolic flow must have zero topological entropy.

REFERENCES

1. H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–50.

2. P. Julius, *A point distal homomorphism with positive topological entropy*, preprint.

3. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

4. N. Markley, *Characteristic sequences*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **30** (1974), 321–342.

5. M. Paul, *Construction of almost automorphic symbolic minimal flows*, General Topology and Appl. **6** (1976), 45–56.

6. K. Petersen and L. Shapiro, *Induced flows*, Trans. Amer. Math. Soc. **177** (1973), 375–390.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742 USA